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Extinction of elastic wave energy due to scattering in a viscoelastic medium

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Abstract

Forward scattering theorem for elastic longitudinal and shear wave scatterings by an arbitrary-shaped three-dimensional object embedded in a viscoelastic medium is derived. It is shown that the formulae for extinction cross-sections of an object in an energy-absorbing medium are formally the same with those of the object in the lossless elastic medium. Numerical calculations are executed for the longitudinal wave scattering in an epoxy matrix by a spherical inclusion with different material properties. The condition of negative extinction is examined with the causality constraint on the viscoelastic medium taken into account. It is found that the negative extinction occurs in the Rayleigh limit when the attenuation of the medium is sufficiently high and, more restrictedly, the wave speed in the object is larger than that in the medium, while it occurs less likely in the high frequency range considered in this paper ($0 < ka < 100$).

Keywords: Scattering; Forward scattering theorem; Extinction cross-section; Viscoelastic materials

1. Introduction

The forward scattering theorem (or the optical theorem) is one of the fundamental results in the scattering theory and is physically and formally common to any type of waves (Feenberg, 1932; van de Hulst, 1949; Jones, 1955; Tan, 1976; Varatharajulu, 1977). The evaluation of extinction (total) cross-sections of a scattering object embedded in an absorbing medium is often demanded, for example, in the analysis of wave propagation in inhomogeneous materials (Niklasson et al., 1981; Beltzer and Brauner, 1987; Biwa, 2001; Kim, 2003). The extinction cross-section is defined as the rate at which energy is abstracted from the incident wave during the process of scattering. It is usually derived applying the principle of energy conservation to the scattering problem. When the surrounding medium is lossless, the total extinguished power is expressed as the sum of the power scattered and absorbed by the object. The scattered power may be

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evaluated either on the surface of a hypothetical sphere enclosing the object (Varatharajulu, 1977; Gubernatis et al., 1977) or on the object surface (de Hoop, 1959; Tan, 1976).

When the medium is absorbing, the energy extinction is not readily obvious since not only the object but also the medium itself participate in the energy extinction process. Even without an object, the incident wave energy is extinguished in the medium. This fact leads to, as was pointed out by Bohren and Gilra (1979), an expression of energy extinction that depends on the size of the hypothetical sphere which is introduced as a purely mathematical device. Consequently, the decomposition of the extinction into contributions of scattering and absorption (in the medium) becomes unclear. Therefore, the argument of the conservation of energy based on the hypothetical sphere seems no longer appropriate to derive the extinction cross-section in the absorbing medium. This fact may mislead to conclusion that an exact expression of the extinction cross-section does not exist in the absorbing medium. However, the definition of the extinction cross-section does not require the energy conservation to hold.

There have been attempts to extend the ordinary extinction cross-section to the case of a cylindrical object in the absorbing medium (Beltzer and Brauner, 1987; Brauner and Beltzer, 1988; Biwa, 2001). Since their formulations still relied on the energy conservation, as shown in (Kim, 2003), their formulae for extinction cross-sections are valid in the low frequency region where the effect of the absorption is less significant.

In this paper, the forward scattering theorem for elastic wave scatterings by an arbitrary-shaped three-dimensional object embedded in an absorbing medium is derived based on a direct mathematical formulation of an actual experiment for measuring optical extinction (van de Hulst, 1949; Bohren and Gilra, 1979; Wymer and Lakhtakia, 1995). Numerical results of extinction cross-sections of a spherical object with different material properties in epoxy matrix are presented. The conditions under which the paradoxical negative extinction phenomenon occurs are investigated.

2. Far-field scattering amplitudes

Consider a three-dimensional inclusion embedded in an infinite homogeneous viscoelastic medium having density ρ and Lamé elastic constants λ and μ. The inclusion may be an empty or fluid-filled cavity or an elastic (or viscoelastic) solid of arbitrary shape. The configuration of the elastic wave scattering by the inclusion is schematically depicted in Fig. 1, and detailed analysis is given in Appendix A.

The displacement and stress fields of an incident time-harmonic ($\exp(-i\omega t)$) plane longitudinal wave propagating in z-direction can be written as

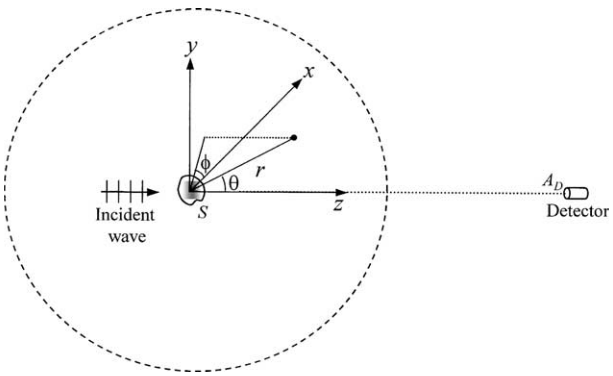

Fig. 1. Schematic showing plane wave scattering by an arbitrary-shaped object, a hypothetical sphere and a detector in the far-field. The detector measures the wave amplitude on its front face without disturbing the wave field.

$$u_z^{\rm inc} = {\rm i}k_{\rm p}\Phi_0 {\rm e}^{{\rm i}k_{\rm p}z}, \tag{1}$$

$$\sigma_z^{\rm inc} = -k_{\rm p}^2\Phi_0(\lambda + 2\mu){\rm e}^{{\rm i}k_{\rm p}z}, \tag{2}$$

where $k_{\rm p}(=\omega/C_{\rm p})$ is the wavenumber associated with the longitudinal wave speed $C_{\rm p} = ((\lambda+2\mu)/\rho)^{1/2}$. Φ_0 which has the dimension of length square is set to be unity for brevity and thus does not appear in the following equations. The scattered displacement vector in the far-field (see Appendix A) is

$$\mathbf{u}^{\rm sca}(\mathbf{r}) \sim \hat{\mathbf{r}} f_{\rm p}(\hat{\mathbf{r}})\frac{{\rm e}^{{\rm i}k_{\rm p}r}}{r} + \left[\hat{\boldsymbol{\theta}} f_{{\rm s}_1}(\hat{\mathbf{r}}) + \hat{\boldsymbol{\varphi}} f_{{\rm s}_2}(\hat{\mathbf{r}})\right]\frac{{\rm e}^{{\rm i}k_{\rm s}r}}{r}, \tag{3}$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ are unit vectors in the spherical coordinate system shown in Fig. 1; $r = |\mathbf{r}| = (x^2+y^2+z^2)^{1/2}$ is the distance from the coordinate origin to a receiver at (x, y, z); $f_{\rm p}(\hat{\mathbf{r}})$, $f_{{\rm s}_1}(\hat{\mathbf{r}})$ and $f_{{\rm s}_2}(\hat{\mathbf{r}})$ are directivity patterns of scattered longitudinal (P) and two mutually orthogonal shear (S_1 and S_2) waves; $k_{\rm s}(=\omega/C_{\rm s})$ is the wavenumber associated with the shear wave speed $C_{\rm s} = (\mu/\rho)^{1/2}$. The wavenumbers are complex-valued, that is, $k_{\rm p,s} = k'_{\rm p,s} + {\rm i}k''_{\rm p,s}$, where imaginary parts are the longitudinal and shear wave attenuation coefficients. Accordingly, the elastic constants and the wave speeds are all complex-valued, and thus frequency-dependent.

The scattered stress dyadic in the far-field is

$$\boldsymbol{\sigma}^{\rm sca}(\mathbf{r}) \sim {\rm i}k_{\rm p}\left[\lambda\mathbf{I} + 2\mu\hat{\mathbf{r}}\hat{\mathbf{r}}\right] f_{\rm p}(\hat{\mathbf{r}})\frac{{\rm e}^{{\rm i}k_{\rm p}r}}{r} + {\rm i}k_{\rm s}\mu\left[(\hat{\mathbf{r}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\mathbf{r}}) f_{{\rm s}_1}(\hat{\mathbf{r}}) + (\hat{\mathbf{r}}\hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{\varphi}}\hat{\mathbf{r}}) f_{{\rm s}_2}(\hat{\mathbf{r}})\right]\frac{{\rm e}^{{\rm i}k_{\rm s}r}}{r}. \tag{4}$$

3. Forward scattering theorems

To derive the forward scattering theorem for the elastic wave scatterings in an absorbing medium, an experimental setup shown in Fig. 1 is considered. This is a simplified schematic of the actual experiment for measuring optical extinction (Bohren and Huffman, 1983) by a single particle or a cluster of particles. A planar detector is located at a large distance from the object in the forward direction such that $k_{\rm p}z \gg 1$ and $k_{\rm s}z \gg 1$. In the elastic wave scattering, it is assumed that the detector measures but does not disturb the wave field. This assumption is not necessary in the optical scattering because the geometrical limit approximation can be applied.

The energy extinction due to whatever reasons such as scattering and/or absorption should appear as a reduction in the amplitude of the incident plane wave after traveling a large distance. Therefore, by comparing intensities of wave fields at the same location far behind the object in the absence and in the presence of the object, the effect (the decrease of the wave intensity) owing solely to the object can be evaluated (van de Hulst, 1949; Bohren and Gilra, 1979). This approach is certainly an alternative to the conventional approach calculating the energy flux out of the surface of the hypothetical sphere, and does not rely on the energy conservation principle. In fact, this approach implements the definition of extinction cross-section in a straightforward manner. van de Hulst (1949) and Bohren and Gilra (1979) used this approach to derive the forward scattering (optical) theorem for the light wave, Wymer et al. (1995) for the acoustic wave in a viscous fluid and Kim (2003) for elastic waves in two-dimensional space.

The average power received by the detector with area $A_{\rm D}$ can be expressed as

$$P_{\rm D} = I_z^{\rm inc} A_{\rm D} + P_{\rm D}^{\rm ext}, \tag{5}$$

where $I_z^{\rm inc}$ and $P_{\rm D}^{\rm ext}$ denote the intensity of the incident wave and the extinguished power at the detector, respectively. Then, the extinction cross-section is obtained by its definition as

$$\Sigma^{\mathrm{ext}} = -\frac{P_{\mathrm{D}}^{\mathrm{ext}}}{I_z^{\mathrm{inc}}}. \tag{6}$$

The average power received by the detector is

$$P_{\mathrm{D}} = \frac{\omega}{2}\mathrm{Im}\int_{A_{\mathrm{D}}} (\sigma_{zj}^{\mathrm{inc}} + \sigma_{zj}^{\mathrm{sca}})\left(\bar{u}_j^{\mathrm{inc}} + \bar{u}_j^{\mathrm{sca}}\right)\mathrm{d}A \tag{7a}$$

$$= \frac{\omega}{2}\mathrm{Im}\left(\int_{A_{\mathrm{D}}} \sigma_{zj}^{\mathrm{inc}}\bar{u}_j^{\mathrm{inc}}\,\mathrm{d}A + \int_{A_{\mathrm{D}}} \sigma_{zj}^{\mathrm{sca}}\bar{u}_j^{\mathrm{sca}}\,\mathrm{d}A + \int_{A_{\mathrm{D}}} \sigma_{zj}^{\mathrm{inc}}\bar{u}_j^{\mathrm{sca}} + \sigma_{zj}^{\mathrm{sca}}\bar{u}_j^{\mathrm{inc}}\,\mathrm{d}A\right) \tag{7b}$$

$$= \int_{A_{\mathrm{D}}} I_z^{\mathrm{inc}}\,\mathrm{d}A + \int_{A_{\mathrm{D}}} I_z^{\mathrm{sca}}\,\mathrm{d}A + \int_{A_{\mathrm{D}}} I_z^{\mathrm{ext}}\,\mathrm{d}A \tag{7c}$$

$$= I_z^{\mathrm{inc}}A_{\mathrm{D}} + P_{\mathrm{D}}^{\mathrm{sca}} + P_{\mathrm{D}}^{\mathrm{ext}}, \tag{7d}$$

where the overbar denotes the complex conjugate.

From Eqs. (1) and (2), the intensity of the incident wave is

$$I_z^{\mathrm{inc}} = \frac{\omega}{2}\mathrm{Im}[\mathrm{i}(\lambda + 2\mu)k_{\mathrm{p}}]\left|k_{\mathrm{p}}\right|^2\mathrm{e}^{-2k_{\mathrm{p}}''z}. \tag{8}$$

From Eqs. (3) and (4), the scattered power at the detector is

$$P_{\mathrm{D}}^{\mathrm{sca}} = \frac{\omega}{2}\mathrm{Im}\int_{A_{\mathrm{D}}} \left(I_r^{\mathrm{sca}}\cos\theta - I_\theta^{\mathrm{sca}}\sin\theta\right)\mathrm{d}A \tag{9a}$$

$$= \frac{\omega}{2}\mathrm{Im}\left[\mathrm{i}(\lambda + 2\mu)k_{\mathrm{p}}\int_{A_{\mathrm{D}}} |f_{\mathrm{p}}(\hat{\mathbf{r}})|^2\frac{\mathrm{e}^{-2k_{\mathrm{p}}''r}}{r^2}\cos\theta\,\mathrm{d}A + \mathrm{i}\mu k_{\mathrm{s}}\int_{A_{\mathrm{D}}} (|f_{\mathrm{s}_1}(\hat{\mathbf{r}})|^2 + |f_{\mathrm{s}_2}(\hat{\mathbf{r}})|^2)\frac{\mathrm{e}^{-2k_{\mathrm{s}}''r}}{r^2}\cos\theta\,\mathrm{d}A\right.$$
$$\left. - \mathrm{i}\int_{A_{\mathrm{D}}}\left(\mu k_{\mathrm{s}} f_{\mathrm{s}_1}(\hat{\mathbf{r}})\bar{f}_{\mathrm{p}}(\hat{\mathbf{r}})\frac{\mathrm{e}^{\mathrm{i}(k_{\mathrm{s}} - \bar{k}_{\mathrm{p}})r}}{r^2} + \lambda k_{\mathrm{p}}\bar{f}_{\mathrm{s}_1}(\hat{\mathbf{r}})f_{\mathrm{p}}(\hat{\mathbf{r}})\frac{\mathrm{e}^{\mathrm{i}(k_{\mathrm{p}} - \bar{k}_{\mathrm{s}})r}}{r^2}\right)\sin\theta\,\mathrm{d}A\right]. \tag{9b}$$

If the detector is observed to be small from the coordinate origin ($\cos\theta \cdot A_{\mathrm{D}} \to A_{\mathrm{D}}$ and $\sin\theta \cdot A_{\mathrm{D}} \to 0$) and the functions in the first two integrals are not oscillatory, Eq. (9b) can be approximated with its leading terms as $z \to \infty$

$$P_{\mathrm{D}}^{\mathrm{sca}} \sim \frac{\omega}{2}\mathrm{Im}[\mathrm{i}(\lambda + 2\mu)k_{\mathrm{p}}|f_{\mathrm{p}}(\hat{\mathbf{z}})|^2\mathrm{e}^{-2k_{\mathrm{p}}''z} + \mathrm{i}\mu k_{\mathrm{s}}(|f_{\mathrm{s}_1}(\hat{\mathbf{z}})|^2 + |f_{\mathrm{s}_2}(\hat{\mathbf{z}})|^2)\mathrm{e}^{-2k_{\mathrm{s}}''z}]\frac{A_{\mathrm{D}}}{z^2}, \tag{10}$$

where $\hat{\mathbf{z}}$ denotes the unit vector in z-direction. At a sufficiently large distance $k_{\mathrm{p,s}}z \gg 1$, the scattered power $P_{\mathrm{D}}^{\mathrm{sca}}$ decreases $\mathrm{O}(z^{-2})$ faster than the power of the incident wave as $z \to \infty$, and therefore it can be neglected. The extinction power due to interference of the incident and scattered waves is

$$P_{\mathrm{D}}^{\mathrm{ext}} = \frac{\omega}{2}\mathrm{Im}\int_{A_{\mathrm{D}}} (\sigma_{zz}^{\mathrm{sca}}\bar{u}_z^{\mathrm{inc}} + \sigma_{zz}^{\mathrm{inc}}\bar{u}_z^{\mathrm{sca}})\mathrm{d}A \tag{11a}$$

$$= \frac{\omega}{2}\mathrm{Im}\left[(\lambda + 2\mu)|k_{\mathrm{p}}|^2\mathrm{e}^{-2k_{\mathrm{p}}''z}\int_{A_{\mathrm{D}}} \frac{\mathrm{e}^{\mathrm{i}k_{\mathrm{p}}(r-z)}}{r}f_{\mathrm{p}}(\hat{\mathbf{r}})\cos^2\theta\,\mathrm{d}A\right.$$
$$- 2\mu k_{\mathrm{s}}\bar{k}_{\mathrm{p}}\int_{A_{\mathrm{D}}} f_{\mathrm{s}_1}(\hat{\mathbf{r}})\frac{\mathrm{e}^{\mathrm{i}(k_{\mathrm{s}}r - \bar{k}_{\mathrm{p}}z)}}{r}\sin\theta\cos\theta\,\mathrm{d}A + 2\lambda|k_{\mathrm{p}}|^2\int_{A_{\mathrm{D}}} f_{\mathrm{p}}(\hat{\mathbf{r}})\frac{\mathrm{e}^{\mathrm{i}(k_{\mathrm{p}}r - \bar{k}_{\mathrm{s}}z)}}{r}\sin^2\theta\,\mathrm{d}A$$
$$\left. - (\lambda + 2\mu)k_{\mathrm{p}}^2\mathrm{e}^{-2k_{\mathrm{p}}''z}\int_{A_{\mathrm{D}}} \bar{f}_{\mathrm{p}}(\hat{\mathbf{r}})\frac{\mathrm{e}^{-\mathrm{i}\bar{k}_{\mathrm{p}}(r-z)}}{r}\cos\theta\,\mathrm{d}A + (\lambda + 2\mu)k_{\mathrm{p}}^2\int_{A_{\mathrm{D}}} \bar{f}_{\mathrm{s}_1}(\hat{\mathbf{r}})\frac{\mathrm{e}^{\mathrm{i}(k_{\mathrm{p}}z - \bar{k}_{\mathrm{s}}r)}}{r}\sin\theta\,\mathrm{d}A\right]. \tag{11b}$$

Note that the following relations are used to represent the scattered displacement and stress fields in the rectangular coordinate system

$$u_z = u_r \cos\theta - u_\theta \sin\theta, \tag{12}$$

$$\sigma_{zz} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - 2\sigma_{r\theta}\cos\theta\sin\theta. \tag{13}$$

The integrals in Eq. (11b) can be commonly in the following form

$$J = \int_{A_{\rm D}} \eta(x,y)\mathrm{e}^{\mathrm{i}kz\xi(x,y)}\,\mathrm{d}A. \tag{14}$$

The asymptotic value of this type of integral can be obtained by applying the method of stationary phase (Born and Wolf, 1987). The only critical point at which the exponential function in the above integral has stationary phase, $\partial\xi/\partial x(x_0,y_0) = 0$ and $\partial\xi/\partial y(x_0,y_0) = 0$ is $(x_0,y_0) = (0,0)$ (or $\hat{\mathbf{r}} = \hat{\mathbf{z}}$) as $z \to \infty$; the leading term of the integral Eq. (14) is

$$J \sim \frac{2\pi \mathrm{i}z}{k}\eta(0,0). \tag{15}$$

Thus, evaluating the integrals in Eq. (11b), the extinction power at the detector is

$$P_{\rm D}^{\rm ext} = \frac{\omega}{2}\mathrm{Im}[\mathrm{i}(\lambda+2\mu)k_{\rm p}]|k_{\rm p}|^2\mathrm{e}^{-2k_{\rm p}''z}\cdot 4\pi\mathrm{Re}\left[\frac{f_{\rm p}(\hat{\mathbf{z}})}{k_{\rm p}^2}\right]. \tag{16}$$

The extinction cross-section is obtained, according to Eqs. (6), (8) and (16), as

$$\Sigma_{\rm p}^{\rm ext} = -4\pi\mathrm{Re}\left[\frac{f_{\rm p}(\hat{\mathbf{z}})}{k_{\rm p}^2}\right]. \tag{17}$$

Those for the shear waves (S_1 and S_2) can be derived in the same manner

$$\Sigma_{\rm s_1,s_2}^{\rm ext} = -4\pi\mathrm{Re}\left[\frac{f_{\rm s_1,s_2}(\hat{\mathbf{z}})}{k_{\rm s}^2}\right]. \tag{18}$$

Eqs. (17) and (18) are the forward scattering theorems generalized for elastic wave scatterings in the absorbing medium and in the same forms, with a rearrangement to account for the complex wavenumbers ($k_{\rm p}$ and $k_{\rm s}$), as those in the lossless elastic medium (Tan, 1976; Varatharajulu, 1977). The same results have been obtained using the integral equations for the scattered elastic wave fields (Kim, submitted). It has been noted by Lim and Hackman (1990) that the additional off-axis term in the expression of extinction cross-section by Gubernatis et al. (1977) is erroneous. Therefore, the ordinary forward scattering theorems holds equally in the absorbing medium. This may be the consequence of the fact that the scattering analysis in an absorbing medium can be formally the same as in the lossless medium if the absorption of the medium is considered with the complex wavenumbers.

The present result is not so surprising in that since the scattered wave forms the shadow in the forward direction and interferes with the incident wave causing the amplitude reduction, the total extinguished power by any possible mechanism, whether the medium is absorbing or non-absorbing, should be proportional to the forward scattering amplitude. The absorption by the object is considered already in the scattered wave amplitude when it is determined through boundary conditions at the object surface.

4. Numerical results and discussion

Numerical calculations have been performed for the longitudinal wave scattering by a spherical inclusion in a highly absorbing viscoelastic epoxy (EPON 828Z) matrix. Fifteen different materials (including the vacuum) that have been considered for the inclusion are categorized into six different groups of materials according to their elastic properties as listed in Table 1. The materials in the same group exhibit a similar behavior. The attenuation in the materials is assumed to increase linearly with frequency, $k''_{\mathrm{p,s}} = m_{\mathrm{p,s}}\omega$. Although this is most appropriate to viscoelastic polymers, the same rule is applied to some metallic materials considered in this paper. The proportionality constants m_{p} and m_{s} are also shown in Table 1. The wave speeds of materials in groups B–D as well as the epoxy matrix are frequency-dependent. The values shown in the Table 1 are wave speeds at $\omega = 0$ ($C_{\mathrm{p}}(0)$ and $C_{\mathrm{s}}(0)$), which are thus real quantities.

Fig. 2 shows normalized extinction cross-sections $\Sigma_{\mathrm{p}}^{\mathrm{ext}}/\pi a^2$ for six representative materials (one from each group) in the normalized frequency range, $0 \leqslant k'_{\mathrm{p0}}a \leqslant 100$, where a denotes the sphere radius and $k'_{\mathrm{p0}} = \omega/C_{\mathrm{p}}(0)$. Aluminum and polystyrene spheres show negative extinction values. One may expect that the negative extinction occur at resonance frequencies of the sphere where the incident wave power is absorbed efficiently producing minima in the scattering power spectrum. However, due to the high rate of energy loss in the matrix, the resonances of the elastic sphere cannot be supported strongly. It is noted that the negative extinction occurs rather at minimum positions of the slowly undulating contribution as shown in Fig. 2. This contribution to the extinction cross-section is due to interference of the waves circumnavigating around the object in opposite directions that is called the background of the scattering object in the resonance scattering theory (Brill and Gaunaurd, 1987). This observation illustrates significance of the absorption of the medium over the scattering effect in extinguishing the wave energy.

Some comments have to be made in comparison with the results of Brauner and Beltzer (1988) for horizontally polarized shear wave scattering by a circular cylinder. Some of their results show negative

Table 1
Material properties used in the calculations

Group[a]	Material	C_{p} ($\omega = 0$) (m/s)	C_{s} ($\omega = 0$) (m/s)	m_{p} (s/m)	m_{s} (s/m)	ρ (kg/m^3)
Matrix	Epoxy (Epon 828Z)	2640	1200	0.73×10^{-5}	0.22×10^{-4}	1202
A	Glass	5280	3240	–	–	2490
	Aluminum	6305	3113	–	–	2690
	Magnesium	5770	3050	–	–	1740
	HPA	7056	3753	–	–	3160
B	Copper	3700	2300	0.93×10^{-6}	3.22×10^{-6}	8230
	Silver	2700	1600	0.12×10^{-6}	0.47×10^{-6}	10 500
	Lead	2210	860	0.38×10^{-5}	0.17×10^{-4}	11 300
	Gold	2000	1200	0.12×10^{-6}	0.47×10^{-6}	19 300
C	TiC	10 000	6200	–	–	4900
	Steel	5940	3220	0.10×10^{-6}	0.41×10^{-6}	7800
	Germanium	5285	3376	–	–	5360
D	Polystyrene	2400	1102	0.60×10^{-6}	0.18×10^{-5}	1050
	PMMA	2669	1305	0.22×10^{-5}	0.07×10^{-4}	1170
E	Water	1460	–	–	–	1000
F	Vacuum	–	–	–	–	–

[a] Characteristics of materials in different groups are A: stiff and light; B: soft and heavy; C: stiff and heavy; D: polymers whose properties are close to those of the matrix.

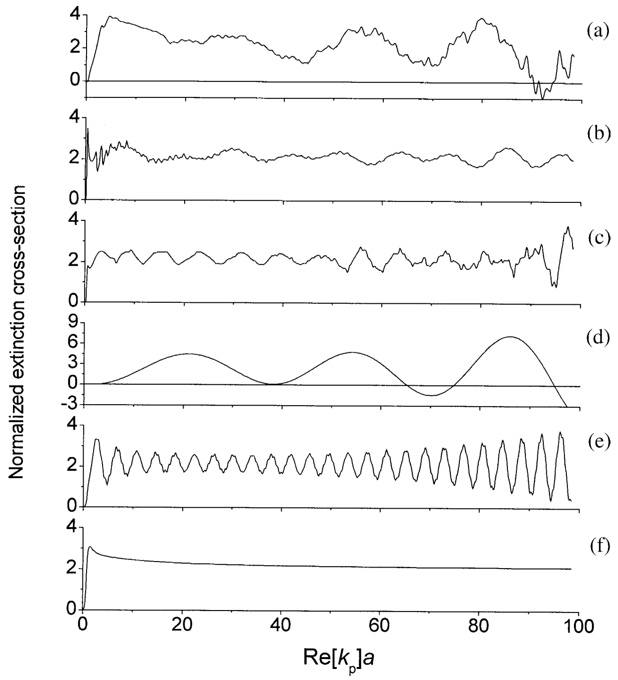

Fig. 2. Extinction cross-sections for longitudinal wave scatterings in the epoxy by a sphere with different material properties. (a) Aluminum, (b) copper, (c) TiC, (d) polystyrene, (e) water, (f) cavity.

extinctions in $5 \leqslant k_s a \leqslant 10$. However, this is unlikely to be so because of the following two reasons: First, the extinction cross-section was underestimated in their theory at frequencies higher than $k_s a = 2$ as shown in Kim (2003); second, the Kelvin–Voight viscoelasticity model used in their calculations gives the attenuation that increases with frequency faster than a linear function. The causality constraint for a material to be physically realizable bounds the attenuation to increase with frequency not as fast as a linear function of ω as $\omega \to \infty$ (Weaver and Pao, 1981). Consequently, the Kelvin–Voight viscoelasticity model produces unphysically high attenuation rate, which resulted in the negative extinction in relatively low frequency region (Brauner and Beltzer, 1988). The linearly increasing attenuation with quite high proportion rate for the epoxy considered in this paper may be regarded as nearly maximum admissible attenuation in real viscoelastic materials. Nonetheless, among fifteen materials that have been simulated, only aluminum, polystyrene and PMMA showed negative extinction in $1 \leqslant k'_{p0}a \leqslant 100$. Therefore, the negative extinction is quite difficult to occur in the elastic medium at high frequencies, whereas it occurs in a wide frequency range in the viscous fluid (Wymer et al., 1995). It is noted that the normalized extinction cross-section of a spherical cavity (Fig. 2(f)) tends to two times the geometrical cross-section in the high frequency limit $k'_{p0}a \to \infty$ just as in the lossless medium. This is analogous to the electromagnetic wave scattering by a perfectly conducting sphere (Wymer and Lakhtakia, 1995).

The normalized extinction cross-sections of the spherical inclusion in the low frequency $k'_{p0}a \leqslant 1$ are plotted in Fig. 3 for the six representative materials. Most of the materials have negative values of the extinction cross-section except for the spherical cavity and the water-filled sphere. It is interesting that this phenomenon occurs well in the low frequency range in which the dynamic effect is usually neglected and thus the attenuation of the medium might have been omitted in the calculation. The polystyrene sphere shows a small negative extinction in this frequency region due to very small difference in material

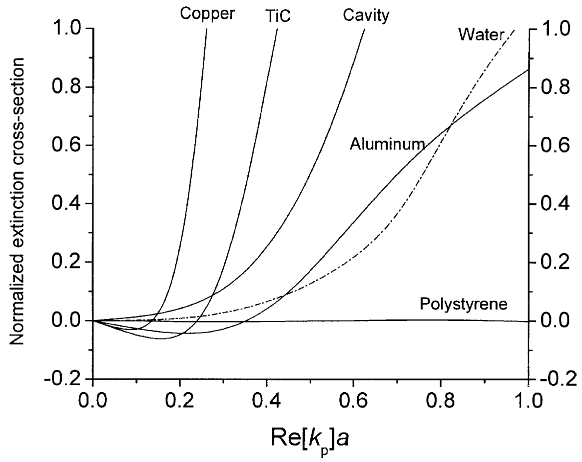

Fig. 3. Extinction cross-sections for longitudinal wave scatterings in the epoxy by a sphere with different material properties in the Rayleigh limit.

properties. The negative extinction seems to be apparently paradoxical but it can be readily understood: In a medium which is more absorbing than the scattering object, it is possible for the detector to receive more energy when the object is present than when it is absent (Bohren and Gilra, 1979). In fact, the negative extinction in the low frequency is because the extinction due to the scattering is smaller than the extinction due to the viscoelastic loss in the same volume that the object occupies.

To find conditions in terms of properties of the constituent materials under which the negative extinction occurs in the Rayleigh limit, the following simulations are performed. The parameters that influence the extinction cross-section are attenuations in the medium and in the object, k''_p/k'_p, k''_s/k'_s, κ''_p/κ'_p, κ''_s/κ'_s; material property contrasts, C_p/c_p, C_s/c_s, ρ_2/ρ_1; ratios between longitudinal and shear waves, C_p/C_s, c_p/c_s; the frequency, $k'_{p0}a$. Here, $\kappa_{p,s}(= \kappa'_{p,s} + i\kappa''_{p,s})$ and $c_{p,s}$ are complex wave numbers and wave speeds in the inclusion. To reduce the number of parameters, the attenuation in the object is ignored. It is also assumed that $k''_p/k'_p = k''_s/k'_s$ and $C_p/C_s = c_p/c_s$. These assumptions do not remove the generality of the simulation since they are fairly reasonable in many materials. The frequency is $k'_{p0}a = 0.05$; the longitudinal wave speed of the surrounding medium is $C_p = 2640$ m/s; $C_p/C_s = 2.2$ at this frequency. Then, the remaining parameters are k''_p/k'_p, C_p/c_p and ρ_2/ρ_1. In Fig. 4, the conditions of the negative extinction in terms of the normalized attenuation and the wave speed ratio are shown for different density ratios. The dark areas are where the extinction cross-section has negative values at this frequency. Note that the condition may vary, but slightly, depending on the frequency. It is interesting to note that the negative extinction occurs mostly when $C_p/c_p < 1$, which is also the reason that the cavity and the water-filled inclusion have non-negative extinctions as shown in Fig. 3. The occurrence conditions depend strongly on the density ratio. For example, the negative extinction can occur only for very high attenuation and $0.3 < C_p/c_p < 1$, when $\rho_2/\rho_1 = 3.0$. When $\rho_2/\rho_1 > 3.0$, the negative extinction does not exist for any material combination. For other density ratios, it occurs in the medium with quite high attenuation, $k''_p/k'_p > 2 \times 10^{-4}$. A full physical understanding of the dependence of the negative extinction on the parameters is quite complicated. However, since the condition depends on relative dominance between the scattering and the absorption and intuitively the scattered wave amplitude is determined by reflectivity (or transmissibility) of the object, the occurrence conditions shown in Fig. 4 will be related closely to these properties of the object. The ratio of acoustic impedances between the matrix and the object ($\rho_2 c/\rho_1 C$) along with k''_p/k'_p may be used in the further study.

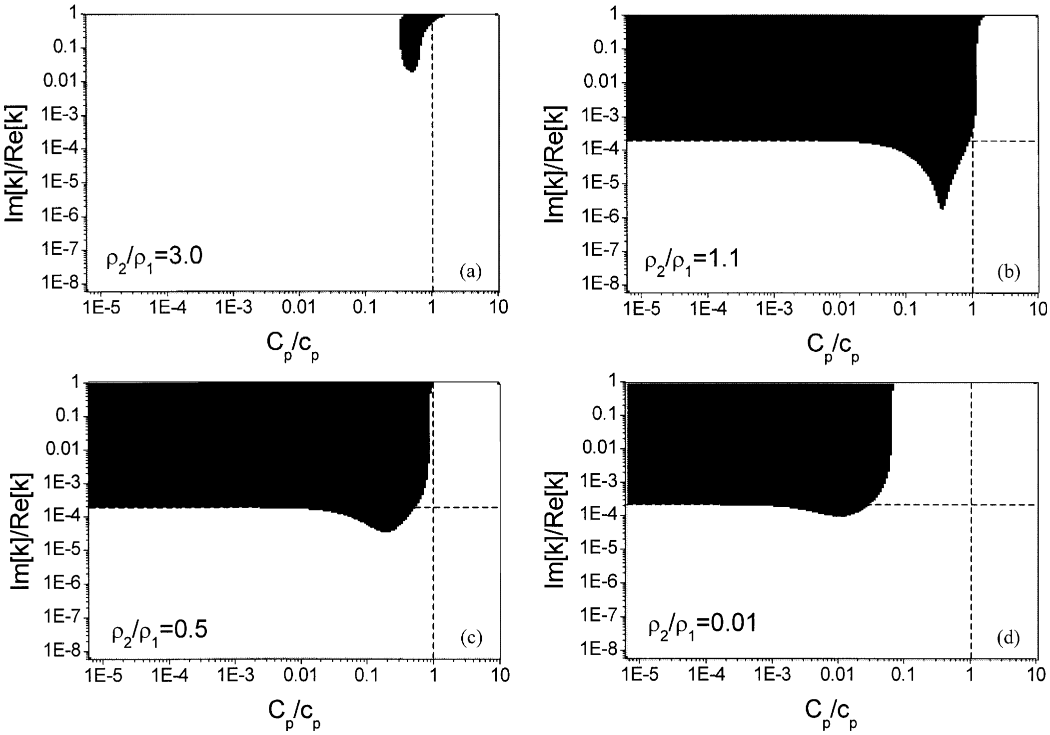

Fig. 4. Conditions for the negative extinction in the Rayleigh limit ($k'_{p0}a = 0.05$). (a) $\rho_2/\rho_1 = 3.0$, (b) $\rho_2/\rho_1 = 1.1$, (c) $\rho_2/\rho_1 = 0.5$, (d) $\rho_2/\rho_1 = 0.01$.

5. Conclusion

The ordinary forward scattering theorem for elastic wave scatterings in the lossless medium is generalized to the case in which the medium is energy-absorbing. The mathematical formulation is inspired by the actual experiment for measuring the optical extinction. The derived expressions of extinction cross-sections are in the same forms with those for the lossless medium. The conditions of the negative extinction are investigated for different materials in different frequency regions. It is found from numerical calculations that the negative extinction occurs in the Rayleigh limit when the attenuation of the medium is sufficiently high and, more restrictedly, the wave speed in the object is larger than that in the medium.

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Appendix A. Elastic wave scattering by a three-dimensional object

The formal analysis of elastic waves scattering by a three-dimensional object has been given by Varatharajulu (1977). Therefore, those equations in (Varatharajulu, 1977) can be directly adopted only with a sign correction. The potentials for the plane time-harmonic incident longitudinal (P) wave propagating in z-direction are written as

$$\Phi^{\mathrm{inc}} = \Phi_0 \mathrm{e}^{\mathrm{i}(k_{\mathrm{p}}z-\omega t)}, \tag{A.1}$$

$$\Psi^{\mathrm{inc}} = 0, \tag{A.2}$$

$$\Pi^{\mathrm{inc}} = 0, \tag{A.3}$$

where ω is the angular frequency. The displacement and stress fields are found by the following relations

$$\mathbf{u} = \nabla\Phi + k_{\mathrm{s}}\nabla \times [\hat{\mathbf{r}}\Psi] + \nabla \times \nabla \times [\hat{\mathbf{r}}\Pi], \tag{A.4}$$

$$\boldsymbol{\sigma} = \lambda\mathbf{I}\nabla \cdot \mathbf{u} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla), \tag{A.5}$$

where $\mathbf{I}$ is the second order identity tensor.

The potentials for the scattered longitudinal wave and two mutually orthogonal shear waves (Morse and Feshbach, 1953) are

$$\Phi^{\mathrm{sca}} = \Phi_0 \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{mn}[A_n Y^{\mathrm{e}}_{nm}(\theta,\phi) + B_n Y^{\mathrm{o}}_{nm}(\theta,\phi)]h_n(k_{\mathrm{p}}r), \tag{A.6}$$

$$\Psi^{\mathrm{sca}} = \Phi_0 \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{mn}[C_n Y^{\mathrm{e}}_{nm}(\theta,\phi) + D_n Y^{\mathrm{o}}_{nm}(\theta,\phi)]h_n(k_{\mathrm{s}}r), \tag{A.7}$$

$$\Pi^{\mathrm{sca}} = \Phi_0 \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{mn}[E_n Y^{\mathrm{e}}_{nm}(\theta,\phi) + F_n Y^{\mathrm{o}}_{nm}(\theta,\phi)]h_n(k_{\mathrm{s}}r), \tag{A.8}$$

where $Y^{\mathrm{e,o}}_{nm}(\theta,\phi)$ are the even and odd spherical harmonics, $h_n(kr)$ is the first kind spherical Bessel function of order n, and A_n, B_n, C_n, D_n, E_n and F_n are unknown scattering coefficients that can be determined from boundary conditions. The normalization constant γ_{mn} is given by

$$\gamma_{mn} = [\varepsilon_n(2n+1)(n-m)!/4\pi(n+m)!]^{1/2}, \tag{A.9}$$

where ε_n is the Neumann factor ($\varepsilon_0 = 1$, $\varepsilon_n = 2$ for $n > 0$). Here, it should be noted that in the Debye scattering theory the scattering fields in Eqs. (A.6)–(A.8) can be assumed in the same forms regardless of whether the surrounding medium is lossless or absorbing.

In the far-field, the spherical Hankel functions can be approximated as

$$h_n(kr) \sim (-\mathrm{i})^{n+1}\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}. \tag{A.10}$$

Then, one obtains the far-field asymptotic scattered displacement field as Eq. (3) and the scattering amplitudes or the directivity patterns are

$$f_{\mathrm{p}}(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{mn}\mathrm{i}^{-n}[A_n Y^{\mathrm{e}}_{nm}(\theta,\phi) + B_n Y^{\mathrm{o}}_{nm}(\theta,\phi)], \tag{A.11}$$

$$f_{\mathrm{s_1}}(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma_{mn}\left[\frac{\mathrm{i}^{-(n+1)}}{\sin\theta}\frac{\partial}{\partial\phi}(C_n Y^{\mathrm{e}}_{nm}(\theta,\phi) + D_n Y^{\mathrm{o}}_{nm}(\theta,\phi)) - \mathrm{i}^{-n}\frac{\partial}{\partial\theta}(E_n Y^{\mathrm{e}}_{nm}(\theta,\phi) + F_n Y^{\mathrm{o}}_{nm}(\theta,\phi))\right], \tag{A.12}$$

$$f_{s_2}(\hat{\mathbf{r}}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma_{mn}\left[\mathrm{i}^{-(n+1)}\frac{\partial}{\partial\theta}(C_nY^{\mathrm{e}}_{nm}(\theta,\phi)+D_nY^{\mathrm{o}}_{nm}(\theta,\phi))+\frac{\mathrm{i}^{-n}}{\sin\theta}\frac{\partial}{\partial\phi}(E_nY^{\mathrm{e}}_{nm}(\theta,\phi)+F_nY^{\mathrm{o}}_{nm}(\theta,\phi))\right]. \tag{A.13}$$

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